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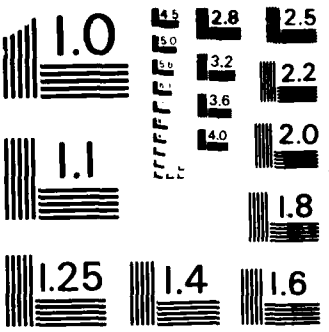

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USING INFLUENCE DIAGRAMS*

by

Richard E. Barlow[1]

ORC 87-1

January 1987

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USING INFLUENCE DIAGRAMS*

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ORC 87-1 January 1987

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ABSTRACT

This paper is to appear in the conference volume on "Accelerated Life Testing and Experts' Opinions in Reliability," C. Clarotti and D. V. Lindley, editors. The conference took place in Lerici, Italy, July 28 to August 1, 1986. This paper is an exposition of the ideas of R. A. Howard, J. E. Matheson, and R. D. Shachter, and is intended primarily for a statistical audience. The influence diagram can be used to model almost any statistical problem of interest. In most cases, it graphically highlights conditional independence relationships implied by the model. Using decision nodes, the influence diagram provides an alternative to the decision tree for Bayesian decision analysis.

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1. INTRODUCTION

Too often in contemplating a statistical analysis, the statistician becomes absorbed with the mathematical details such as particular probability densities, sufficient statistics, and computational methods, without fully understanding the problem at hand. Influence diagrams provide a valuable aid for modeling the logical and statistical dependencies between random quantities and also between these quantities and decision alternatives. This can be done, initially, without specifying particular probability functions.

The influence diagram is a graphical representation of the relationships between random quantities which are judged relevant to a real problem. In this respect it is a modeling tool. The probability weights on the graph nodes are also based on judgements relevant to a real problem. Although the influence diagram is an abstract mathematical model, its validity is based on some real problem of interest.

Influence diagrams were developed as a Bayesian computer aided modeling tool by Howard and Matheson [1984]. Influence diagrams also provide a graphical representation of conditional independence for random quantities. An algorithm for solving Bayesian decision analysis problems through influence diagram manipulations was constructed by R. Shachter [1986]. This algorithm is useful in statistical applications, especially in experimental design [e.g. Barlow and Zhang (1986)]. The influence diagram is also a generalization of the fault tree representation used in engineering risk analysis.

The influence diagram is a modeling alternative to the decision tree. Since the order of event expansion required by the decision tree is rarely the natural order in which to assess the decision maker's information, the influence diagram may be more useful. For an excellent introduction to decision analysis and decision trees see Lindley [1985].

This paper is largely tutorial since the basic ideas were developed elsewhere. However some of the proofs are original. We begin the discussion with influence diagrams having only probabilistic nodes. Deterministic and decision nodes are treated in later sections.

2. PROBABILISTIC INFLUENCE DIAGRAMS

A probabilistic influence diagram is first of all a directed acyclic graph. Circle nodes in the graph denote random quantities. Arcs joining circle nodes denote possible statistical dependence. Associated with each circle node is a conditional probability function. Conditioning is only with respect to immediate predecessor nodes and is indicated by the direction of the arrows. In Figure 2.1, z is an immediate predecessor of both x and y. From the graph in Figure 2.1, the joint probability function, $p(x,y,z)$, is $p(z)p(x|z)p(y|z)$. Hence from the graph, the random quantity x is conditionally independent of y given z. Note that there is no arc joining nodes x and y. Absence of an arc between two nodes indicates that the node random quantities are conditionally independent given the state of immediate predecessor nodes. This will be proved later.

The influence diagrams in Figures 2.1 and 2.2 are fundamentally different. From the graph in Figure 2.2 the joint probability function

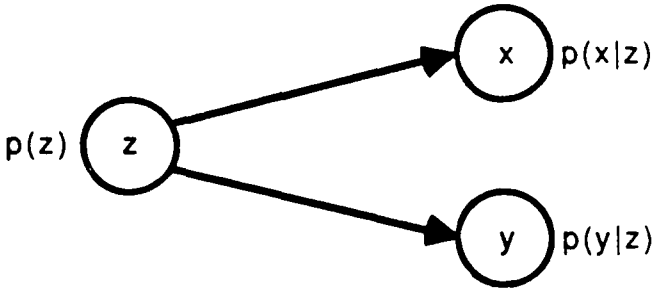

Figure 2.1
x and y Conditionally Independent
Given z

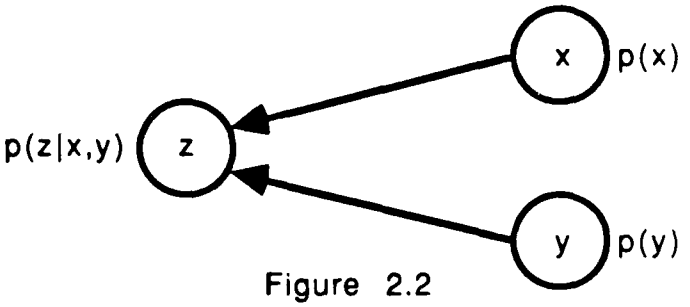

Figure 2.2
x and y Independent

$p(x,y,z)$ is $p(x)p(y)p(z|x,y)$. Hence, in this case, random quantities x and y are unconditionally independent. As we show later, nodes without input arcs are always unconditionally independent.

Given a directed acyclic graph together with node conditional probabilities, there exists a unique joint probability function corresponding to the random quantities represented by the nodes of the graph. This is because a directed graph is acyclic if and only if there exists a list of the nodes such that any successor of a node i in the graph follows node i in the list as well.

Relative to the acyclic graph in Figure 2.3, we can find a list ordering which agrees with the graph ordering. Since the graph is acyclic, at least one node must be a root node (i.e., have no inward pointing arcs). Since x_1 is such a node, let x_1 be the first node in the list ordering. Now delete node x_1 and all arcs incident to x_1 from the graph. Again, the remaining graph must be acyclic and have at least one root node. Clearly x_3 is such a node. Proceeding in this way we obtain the list ordering

$$x_1 < x_3 < x_2 < x_4$$

which agrees with the graph ordering. (The list ordering, however, is not in general unique). It follows that the joint probability function for x_1, x_2, x_3, x_4 can be calculated as

$$\begin{aligned} p(x_1,x_2,x_3,x_4) &= p(x_1)p(x_3|x_1)p(x_2|x_1,x_3)p(x_4|x_1,x_2,x_3) \\ &= p(x_1)p(x_3|x_1)p(x_2|x_1,x_3)p(x_4|x_2,x_3) \end{aligned}$$

since, from the graph, x_4 depends only on x_2 and x_3. If the graph were not acyclic, the conditional probability weights on the graph nodes would not determine the joint probabililty function of the node random quantities.

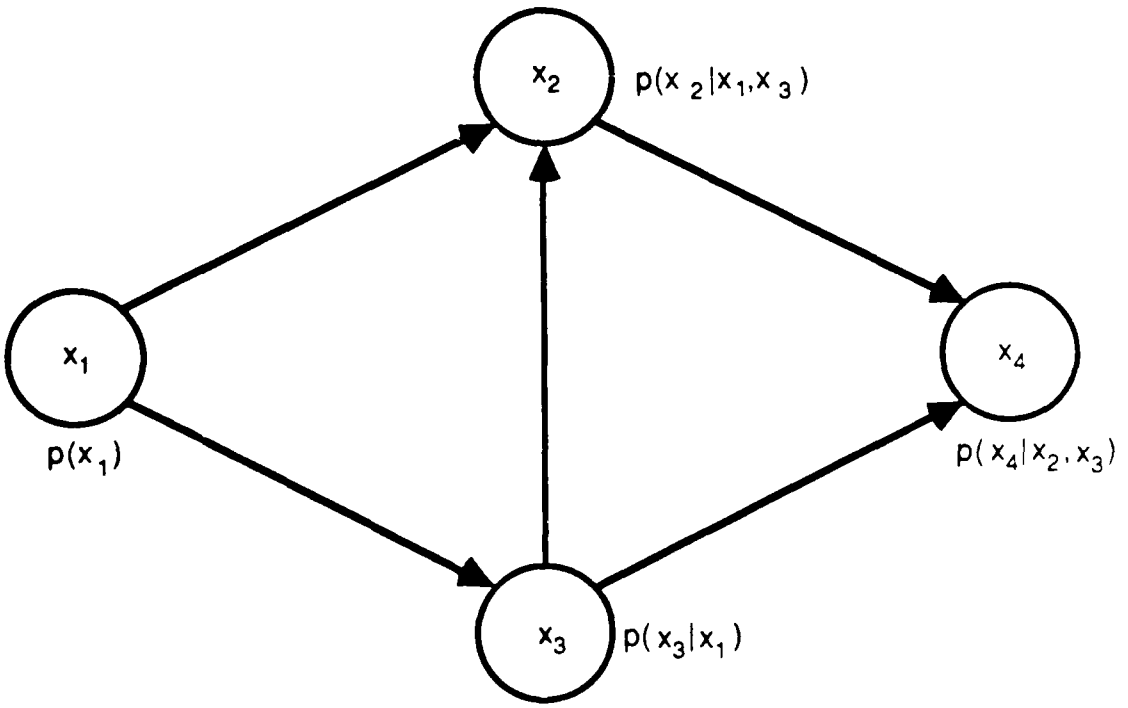

Figure 2.3
A Bridge Type Influence Diagram

3. CONDITIONAL INDEPENDENCE

We will need the concept of a directed path. By a directed path from node x_i to node x_j we mean a chain of ordered pairs (x_i, x_{k_1}), (x_{k_1}, x_{k_2}), ..., (x_{k_t}, x_j) corresponding to directed arcs which lead from x_i to x_j. If there is no path (directed or undirected) in an influence diagram from one node to another, then the corresponding random quantities are unconditionally independent. In graph theory terminology, the graph is disconnected and the nodes are in separate components.

In Figure 3.1, ovals are used to denote disjoint sets of immediate graph predecessor nodes. Thus w_i denotes a set of immediate predecessor nodes to x_i which are not also immediate predecessors of x_j, while w_{ij} denotes a set of nodes which are immediate predecessors of both x_i and x_j.

Theorem 3.1. In an influence diagram, if there is no arc from x_i to x_j nor is there an arc from x_j to x_i, then x_i and x_j are conditionally independent given immediate predecessor nodes; i.e.,

$$p(x_i, x_j | w_i, w_j, w_{ij}) = p(x_i | w_i, w_{ij})\, p(x_j | w_j, w_{ij}).$$

Proof. Since an influence diagram is acyclic, there exists a list ordering which can be used to calculate the joint distribution of node random quantities; i.e.,

$$p(x_1, x_2, \ldots, x_n) = \ldots p(x_i | w_i, w_{ij}) \ldots p(x_j | w_j, w_{ij}) \ldots .$$

Fix x_i, x_j, w_i, w_j, w_{ij}, divide by $p(w_i, w_j, w_{ij}) > 0$ and integrate with respect to all other random quantities. Since x_i and x_j do not belong to (w_i, w_j, w_{ij}) we have

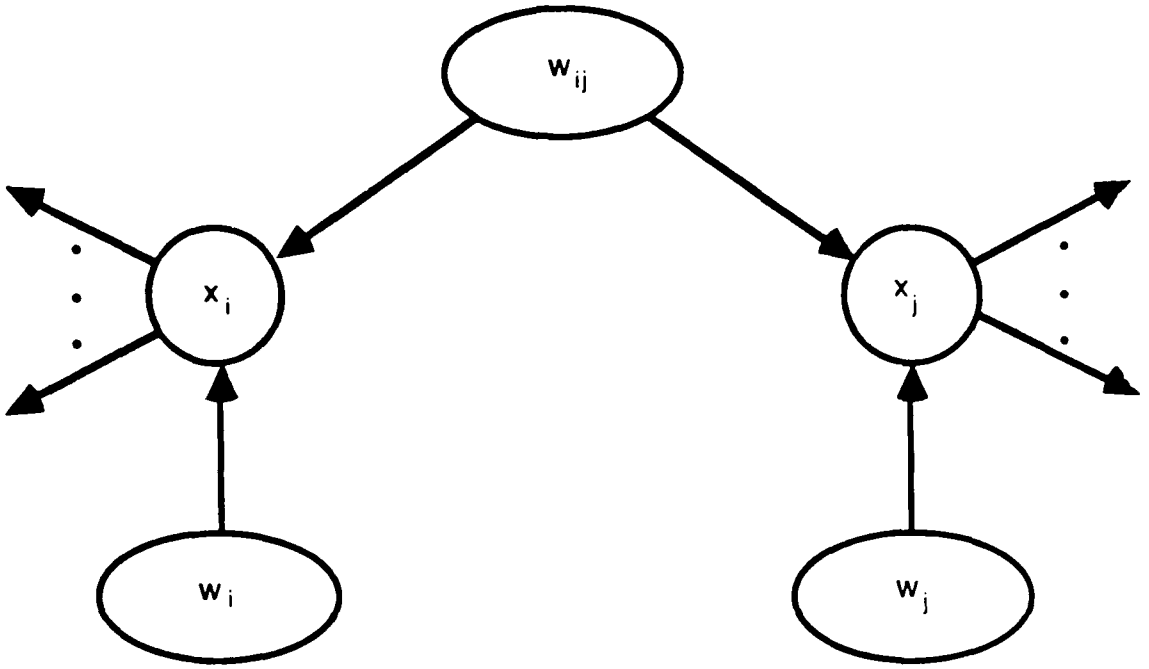

Figure 3.1

Conditional Independence

$$p(x_i, x_j | w_i, w_j, w_{ij}) = p(x_i | w_i, w_{ij})\ p(x_j | w_j, w_{ij}).$$

QED

In Figure 2.3, x_1 and x_4 are conditionally independent given x_2 and x_3 since x_1 has no predecessors and x_2 and x_3 are immediate predecessors of x_4.

Corollary 3.2. In an influence diagram, if nodes x_i and x_j have no incoming arcs, i.e., (w_i, w_{ij}, w_j) is empty, then x_i and x_j are unconditionally independent.

Proof. Since x_i and x_j have no incoming arcs, they have no connecting arc and Theorem 3.1 applies. Since (w_i, w_{ij}, w_j) is empty, x_i and x_j are unconditionally independent.

QED

Although the probabilistic influence diagram is a useful device for illustrating independence, it may not graphically display all the independence relationships implied by the joint probability function. We can use Figure 2.2 again to illustrate these ideas. Suppose a crime has been committed and z denotes the blood type of a blood stain found at the scene of the crime. A suspect is in hand. Let y be the suspect's blood type. For convenience we consider only two blood types, say type 1 and type 2. Let $p(y=1) = \theta$ where θ is the frequency of blood type 1 in the general population.

Let

$$x = \begin{cases} 1 & \text{if the suspect is guilty} \\ 0 & \text{otherwise.} \end{cases}$$

Let $p(z=1 \mid y=1, x=1) = 1$

and $p(z=1 \mid y, x=0) = \theta.$

Let $p(x=1)$ be our prior probability that the suspect is guilty.

As Figure 2.2 shows, we have judged x and y independent, a priori, that is, the guilt or innocence of the suspect is independent of his/her blood type. However, if the suspect is guilty and his/her blood type is 1, then we have judged that the blood stain found at the scene of the crime must also be type 1 so that x, y and z are not independent. But we have also judged x and z independent and this is not obvious from the graph. It is only clear after we do the probability calculations. (However, it can be shown that y and z are dependent in this case.)

4. BAYES' THEOREM

Two probabilistic influence diagrams with the same nodes and the same joint probability function for random quantities will be said to be equi alent. Under certain conditions we can reverse the arc between two nodes, add arcs and, after replacing the node conditional probability weights by appropriate new conditional probabilities, the two probabilistic influence diagrams will be equivalent.

Theorem 4.1. In an influence diagram [cf. Figure 4.1] suppose there is an arc from node x_i to x_j but no other directed path from node x_i to node x_j. If

1. arcs from w_i to x_j and from w_j to x_i are added;
2. the arc from x_i to x_j is reversed;
3. $p(x_j|x_i,w_j,w_{ij})$ is replaced by
$$p(x_j|w_i,w_j,w_{ij}) = \int p(x_j|x_i,w_j,w_{ij})\, p(x_i|w_i,w_{ij})\, dx_i > 0;$$

and 4. $p(x_i|w_i,w_{ij})$ is replaced by
$$p(x_i|x_j,w_i,w_j,w_{ij}) = p(x_j|x_i,w_j,w_{ij})\, p(x_i|w_i,w_{ij}) / p(x_j|w_i,w_j,w_{ij})$$

(i.e., Bayes' theorem) then the probabilistic influence diagrams are equivalent; i.e., the influence diagrams in Figures 4.1 and 4.3 are equivalent.

Proof. Since there is only one directed path from x_i to x_j, namely the arc from x_i to x_j, we can add arcs from w_i to x_j and from w_j to x_i without creating any cycles. This will only create pseudo dependencies. Clearly the influence diagram corresponding to the augmented graph, Figure 4.2, is equivalent to the original influence diagram.

Consider a list ordering for the augmented graph. Clearly x_i must precede x_j in this list ordering, and in fact

$$\ldots < (w_i, w_j, w_{ij}) < x_i < x_j < \ldots$$

since there is no other directed path from x_i to x_j; i.e., there can be no other nodes between x_i and x_j in the list ordering.

We need only show that the joint probability function for the transformed influence diagram in Figure 4.3 is the same as that for the original influence diagram. The joint probability function for the original influence diagram based on the list ordering for the augmented influence diagram will contain the two terms

$$p(x_i|w_i, w_{ij})\ p(x_j|x_i, w_j, w_{ij}). \tag{4.1}$$

If we both multiply and divide (4.1) by

$$p(x_j|w_i, w_j, w_{ij}) = \int p(x_j|x_i, w_j, w_{ij})\ p(x_i|w_i, w_{ij})\ dx_i > 0$$

then (4.1) becomes

$$p(x_i|x_j, w_i, w_j, w_{ij})\ p(x_j|w_i, w_j, w_{ij})$$

by Bayes' theorem. The joint probability functions are equal since all the other terms in the two joint probability functions are the same.

QED

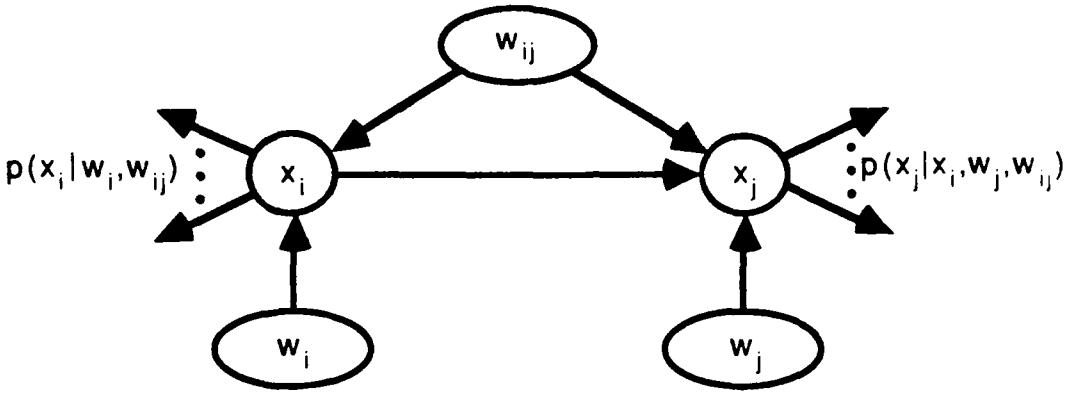

Figure 4.1

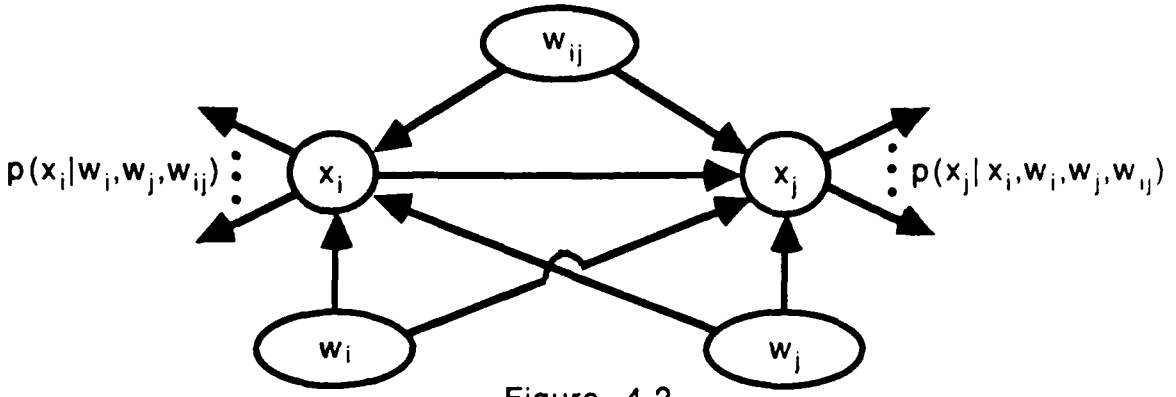

Figure 4.2

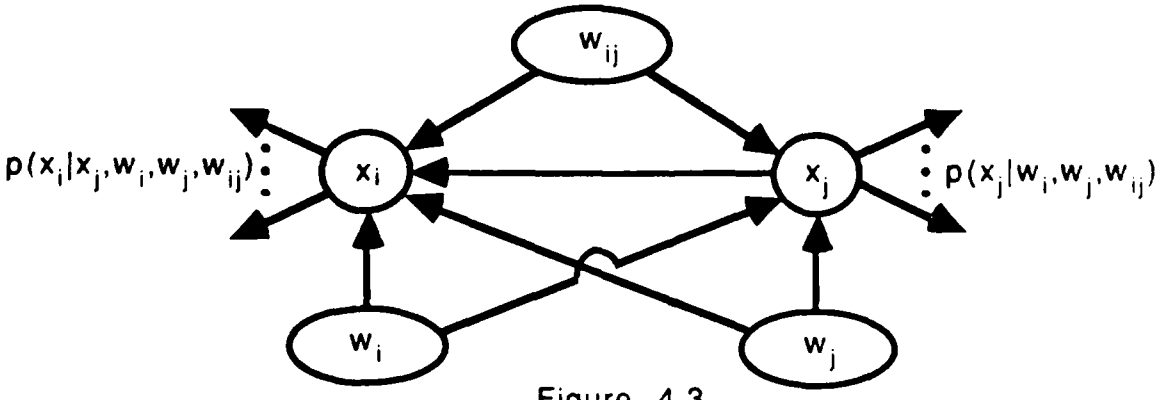

Figure 4.3

Arc reversal corresponds to applying Bayes' theorem. However it also involves the addition of arcs. These arcs may, in some cases, involve only pseudo dependencies. In this sense information may have been lost as a result of arc reversal. For these reasons, we want to avoid arc reversal whenever possible.

5. BARREN NODES

Consider Figure 5.1 and the problem of calculating $p(x_1|x_5)$. Distinguish nodes x_1 and x_5 by shading them as in Figure 5.1. Relative to our problem, we claim that node x_4 is irrelevant. That is, relative to our problem, we can delete node x_4 and all arcs incident to node x_4. The reason we may do this is that in any list ordering for the influence diagram, node x_4 may always be listed last; i.e.,

$$x_1 < x_3 < x_2 < x_5 < x_4 ,$$

and we can calculate the joint probability function

$$p(x_1, x_5)$$

without reference to node x_4. Hence, x_4 is <u>noninformative</u> relative our problem.

In general let x_J $[x_K]$ denote a vector of nodes

$$\{x_i \mid i \in J\} \quad [\{x_i \mid i \in K\}]$$

where $J \cap K = \phi$. If we are interested in calculating $p(x_J|x_K)$ or $p(x_K|x_J)$ we first distinguish nodes x_i where $i \in J \cup K$.

<u>Definition</u>. (Barren Node). In an influence diagram with nodes $\{x_i \mid i \in J \cup K\}$ distinguished and $J \cap K = \phi$, we say that node x_b is barren with respect to the distinguished nodes if

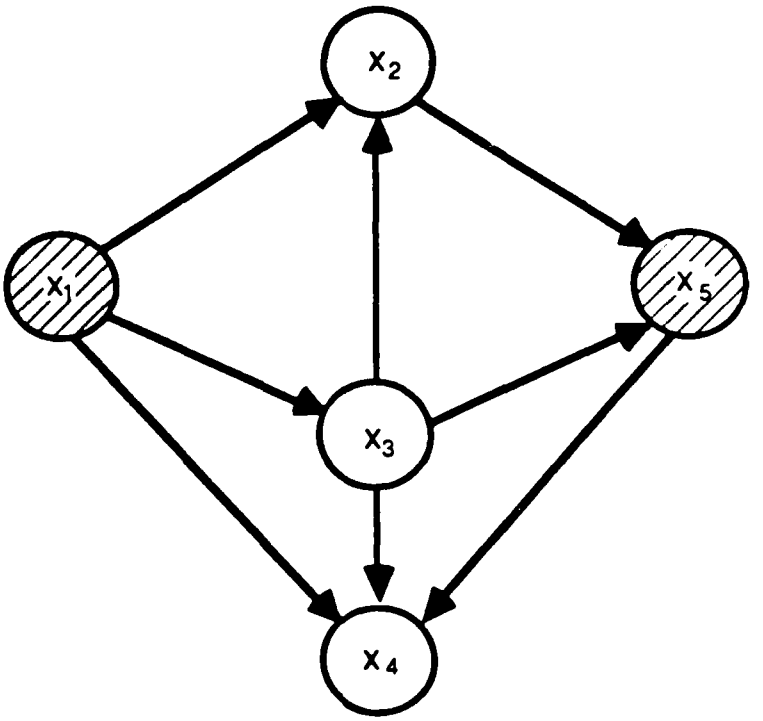

Figure 5.1

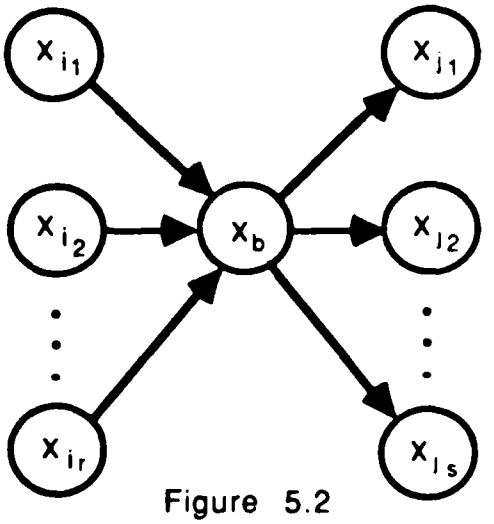

Figure 5.2

1. $b \notin J \cup K$;

and 2. all arcs incident to x_b point toward x_b.

Theorem 5.1. Given a probabilistic influence diagram with distinguished nodes $\{x_i \mid i \in J \cup K\}$ and any node x_b, $b \notin J \cup K$, we can always find an equivalent influence diagram in which node x_b is barren with respect to $J \cup K$.

Proof. Figure 5.2 represents node x_b in the influence diagram where

$$x_{i_1} < x_{i_2} < \ldots < x_{i_r}$$

are the immediate predecessor nodes of x_b and

$$x_{j_1} < x_{j_2} \ldots < x_{j_s}$$

are the immediate successor nodes of x_b. Node x_b will become barren if we can reverse all the arcs from x_b to successor nodes without changing the joint probability function $p(x_i \mid i \in J \cup K)$.

By Theorem 4.1 we can first reverse the arc from x_b to x_{j_1} since this cannot introduce a cycle. (Note that were we to reverse the arc from x_b to x_{j_2} first we could possibly introduce a cycle.) Proceeding in this way we will, by Theorem 4.1, eventually have an equivalent influence diagram in which x_b is barren with respect to $J \cup K$.

QED

6. AN ALGORITHM FOR CALCULATING $p(x_K|x_J)$

To calculate $p(x_K \mid x_J)$ we first distinguish nodes $\{x_i \mid i \in J \cup K\}$. Let x_B be the vector of nodes which are neither predecessors nor successors of (x_J, x_K). (A node, x_i, is a predecessor (successor) of another node, x_j,

if there is a directed path from x_i (x_J) to x_j (x_i). Let x_{Pred} denote the vector of predecessor nodes to (x_J, x_K) and x_S the vector of successor nodes to (x_J, x_K). Then there exists a list ordering for the influence diagram such that

$$x_{Pred} < (x_J, x_B, x_K) < x_S.$$

Clearly x_S are barren relative to our problem and may be deleted. Hence we need only consider the remaining graph with x_S and the arcs incident to x_S deleted.

By Theorem 5.1 we can convert all nodes in x_B to barren nodes in such a way that the resulting influence diagram will have the same joint probability function with respect to (x_{Pred}, x_J, x_K) as did the original influence diagram.

To calculate $p(x_K | x_J)$ we need only integrate the joint probability function for (x_{Pred}, x_J, x_K), namely

$$p(x_{Pred})\ p(x_J, x_K | x_{Pred})$$

with respect to x_{Pred} to obtain $p(x_J, x_K)$ from which we can calculate $p(x_K | x_J)$

The suggested algorithm is in the nature of an existence theorem in the sense that $p(x_K | x_J)$ can be found by graphical and probabilistic manipulations using the concept of arc reversal. However an efficient algorithm for doing this is the subject of current research efforts.

7. DECISION NODES AND VALUE NODES

Decision nodes and value nodes are special. A decision node, represented by a rectangle, denotes a decision function, d_i, mapping the

states of immediate input nodes into a set of decision alternatives $\{d_{i1}, d_{i2}, \ldots, d_{im}\}$. (We will often use d_i to represent the decision function as well as the decision taken.) The states of immediate input nodes to a decision node constitute the information available at the time of decision. The decision taken depends on the states of immediate predecessor nodes. Decision nodes introduce a time ordering into the influence diagram which was not present in the probabilistic influence diagram. If decision node d_i precedes decision node d_j in the graph, then decision d_i must be taken before decision d_j. Neither the arcs input to a decision node nor the arcs output from a decision node can be reversed.

Figure 7.1 is an influence diagram with two decision nodes, d_1 and d_2, a value node, v, and probability nodes x, y, z and θ. [This is the influence diagram for the calibration experimental design problem considered in Barlow, Mensing and Smiriga (1986)]. At the time of decision d_2, the decision d_1 which was taken as well as the states of probability nodes y and x are known. The state of probability node z is not known at the time of decision d_1 or decision d_2 since it is not an immediate predecessor node of either decision node. An influence diagram with decision and value nodes must be acyclic. For example, if there were a directed path from decision node d_1 back to d_1, this would imply that we could foretell the future since decision d_2 occurs after decision d_1.

A value node, v, represented by a diamond in Figure 7.1, is a deterministic function of the states of immediate predecessor nodes. A value node has only inward pointing arcs. In Figure 7.1, $v(\cdot,\cdot)$ depends on the decision, d_2, taken as well as the state of probability node z. The

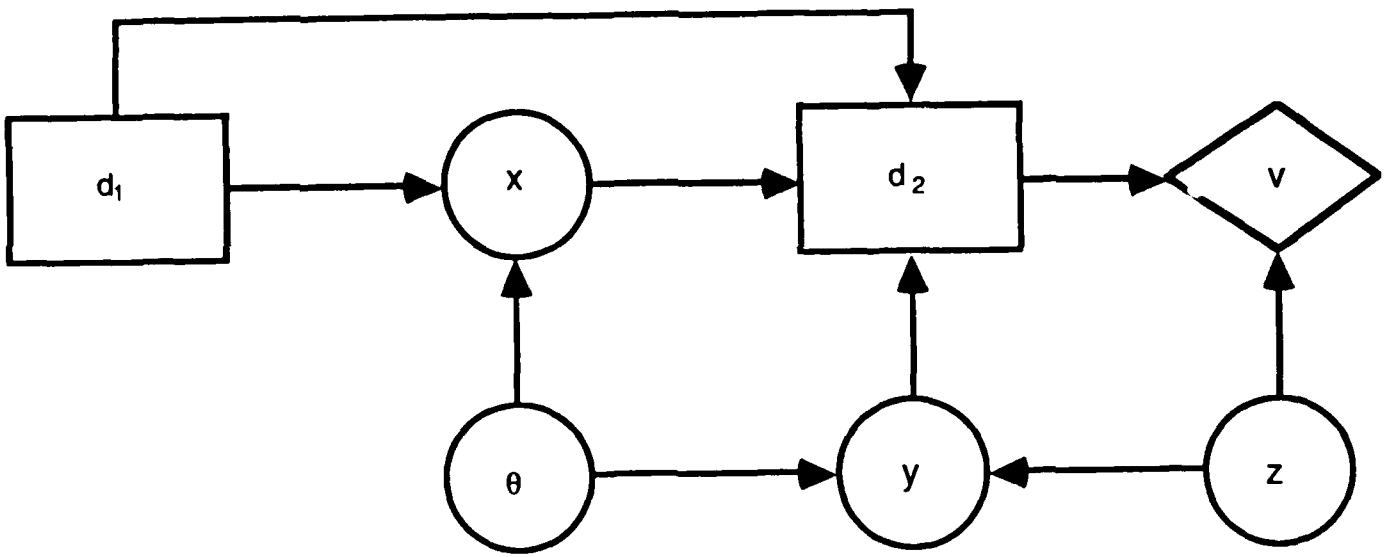

Figure 7.1

optimal decision functions, d_1 and d_2, will depend on the value node. In this sense the value node corresponds to the objective (or utility) function for a decision problem.

8. NODE REDUCTION AND DECISION ANALYSIS

By node reduction, we mean the elimination of a node and its incident arcs in an influence diagram. As we have seen, the reduction of a probability node is valid with respect to distinguished nodes if the probability node is not distinguished and if it is barren (i.e., all incident arcs point inward). The optimal decisions relative to a decision problem can be found by using node reduction and then by maximizing (or minimizing if the value node corresponds to a loss function) a related value function.

A list ordering for the influence diagram in Figure 7.1 is

$$d_1 < z < \theta < x < y < d_2 < v.$$

In Figure 7.1, with decision alternatives d_1 and d_2 specified, the joint probability function for random nodes x, y, z and θ is

$$p(x,y,z,\theta \mid d_1,d_2) = p(z)\ p(\theta)\ p(x \mid d_1,\theta)\ p(y \mid \theta,z).$$

To determine the optimal decision function, d_2, in Figure 7.1 we must first express the value function, $v(\cdot)$, as a function only of d_2 and the states of immediate predecessor nodes to d_2, namely d_1, x and y. The steps in our solution algorithm are as follows:

1. fix the state of all immediate predecessor nodes for all decision nodes;

2. reduce all nodes which are predecessors of v (not just immediate predecessor nodes) that are not decision nodes and are not immediate predecessors of decision nodes;

3. maximize the resulting value function with respect to possible decision alternatives d_2 (which of course may depend on d_1, x and y).

In Figure 7.1, z is a special probability node since it is an immediate predecessor of v. To reduce a probabilistic predecessor of a value node, we must first reverse all other output arcs. This is of course with the proviso that it has no output arcs to decision nodes since these cannot be reversed. We can always do this since the value node may always appear last in an ordered list. Figures 8.1 and 8.2 illustrate the reduction process.

Figures 8.3-8.5 illustrate the decision solution process for a subset of the influence diagram in Figure 7.1. Nodes d, v and y are distinguished. Nodes θ and z are reduced leaving only the distinguished nodes. The reduction process does not change the marginal distribution of y. Finally, for a given value of y, we find a value of d for which

$$\int v(d,z)\ p(z|y)dz$$

is maximum. This value for d is our optimal decision.

CONCLUDING REMARKS

The influence diagram can be used to model almost any statistical problem of interest. In most cases, it graphically highlights conditional independence relationships implied by the model. Using decision nodes, the influence diagram provides an alternative to the decision tree for Bayesian decision analysis.

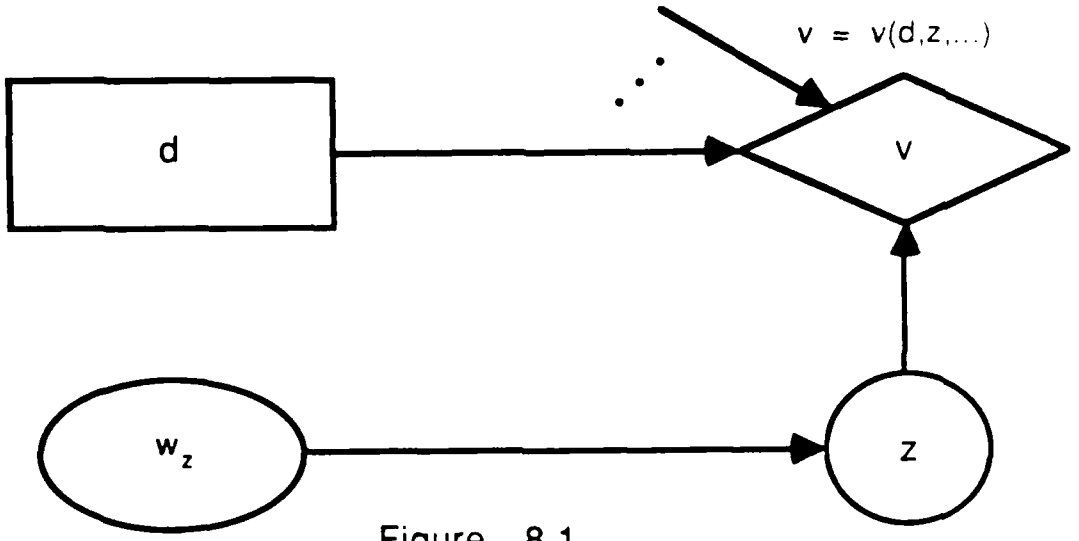

Figure 8.1

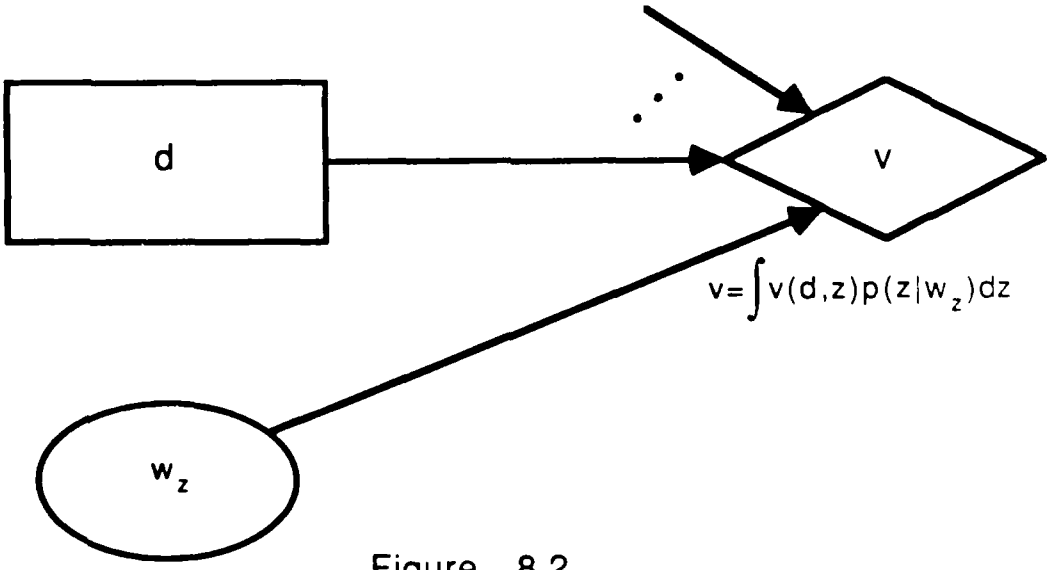

Figure 8.2

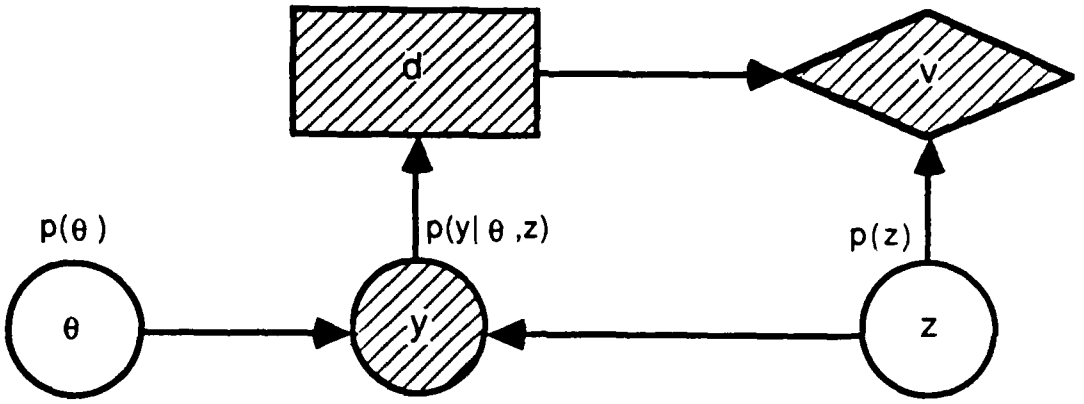

Figure 8.3

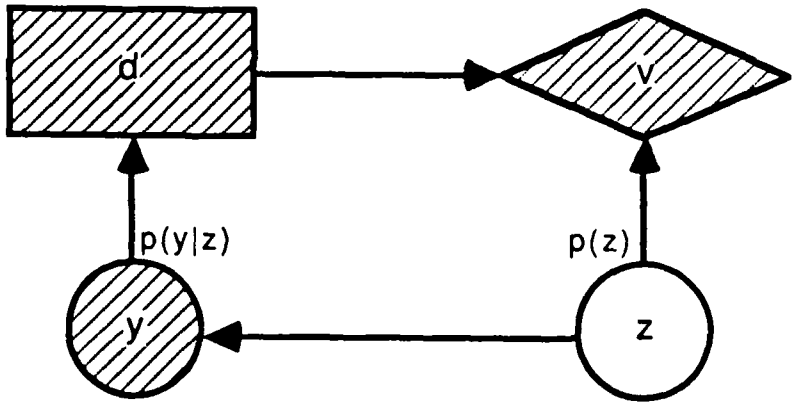

Figure 8.4

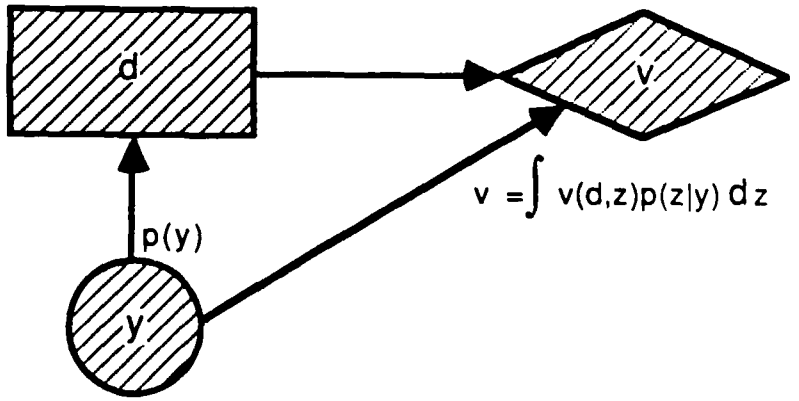

Figure 8.5

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